ON THE TOTAL COLORING OF CERTAIN GRAPHS

BY

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ABSTRACT

In this paper we investigate the total chromatic number of certain graphs. In particular, we show that every cubic graph is totally colorable in five colors.

1. Introduction. Two types of coloring are usually associated with graphs, vertex coloring and edge coloring. A third natural coloring, the total coloring, was introduced by M. Behzad and defined as follows:

DEFINITION A *total coloring* of a graph G is a coloring of the edges and vertices of G such that no two adjacent elements have the same color. (Adjacent elements are two vertices connected by an edge, two edges with a common vertex, or a vertex and an edge containing it). The *total chromatic* number $\tau(G)$ of G is the least number of colors needed for a total coloration of G. In a given coloring of G, $\chi(v)$ will denote the color assigned to v (v can be a vertex or an edge). We denote by $\nu(G)$ the greatest valence among all vertices of G. The complete graph on n vertices will be denoted by C_n. The following interesting conjecture due to M. Behzad relates $\tau(G)$ and $\nu(G)$:

CONJECTURE. (Behzad) $\tau(G) \leqq \nu(G) + 2$.

It is interesting to compare this conjecture with the well known estimates for the chromatic number $cr(G)$ of G (Brooks [2]) and $\varepsilon(G)$, the edge chromatic number (Vizing [3]). These estimates are also given in terms of $\nu(G)$ and are:

$$(1) \quad cr(G) \leqq \nu(G) \quad G \neq C_n \quad (2) \quad \varepsilon(G) \leqq \nu(G) + 1$$

It is easy to give examples of graphs for which $\tau(G) = \nu(G) + 2$, so that Behzad's conjecture, if true, is best possible. The following results obtained by M. Behzad, G. Chartrand and J. K. Cooper Jr. [1] Support this conjecture

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$$\tau(C_n) = \begin{cases} n+1 & n \text{ even} \\ n & n \text{ odd} \end{cases}$$

$$\tau(K_{m,n}) \leqq \max\{m, n\} + 2$$

($K_{m,n}$ is the complete m-by-n bipartite graph). Since every bipartite graph G is $\nu(G)$-edge-colorable, obviously, $\tau(G) \leqq \nu(G) + 2$ for each bipartite graph.

In Section 2 we investigate the total chromatic number of certain families of complete k-partite graphs and establish Behzad's conjecture for cubic graphs.

2. By a complete k-partite graph we mean a graph whose vertex set consists of k disjoint sets of vertices with two vertices joined by an edge iff they belong to distinct sets.

THEOREM 1. *If G is a complete 3-partite graph, then*

$$\tau(G) \leqq \nu(G) + 2$$

PROOF. Let the three independent sets of vertices of G be $A = \{a_1, \cdots, a_k\}$ $B = \{b_1, \cdots, b_m\}$ $C = \{c_1, \cdots, c_n\}$ $k \leqq m \leqq n$. If $k < m$, we can add $m - k$ vertices to A, obtaining a complete 3-partite graph G' with $\nu(G') = \nu(G)$, that contains G as a subgraph. Hence we may assume that $k = m$. If $n = m$ or $n > m + 1$, we color the vertices in A, B and C by $1, 2$ and 3 respectively, and color the following edges as follows

$$\chi(a_i b_i) = 3, \quad \chi(b_i c_i) = 1, \quad \chi(a_i c_i) = 2 \qquad (i \leqq i \leqq m).$$

Obviously, this coloring is admissible. By removing these edges from G, we obtain a graph H with $\nu(H) = \nu(G) - 2$. Hence by Vizing's theorem $\varepsilon(H) \leqq \nu(H) + 1 = \nu(G) - 1$ and therefore

$$\tau(G) \leqq 3 + \varepsilon(H) \leqq \nu(G) + 2.$$

If $n = m + 1$, we use the coloring scheme for the vertices and edges as above for $1 \leqq i \leqq m - 1$. In addition we color

$$\chi(a_m b_m) = 3, \quad \chi(b_m c_m) = 1 \text{ and } \chi(a_m c_{m+1}) = 2.$$

Again the graph H obtained by removing the edges thus colored will have the same properties as in the proceeding case, and the theorem is proved.

A complete balanced k-partite graph is a graph in which all k sets contain the same number of vertices. For such graphs we have

THEOREM 2. *If G is a complete balanced k-partite graph then,*

$$\tau(G) \leqq v(G) + 2.$$

PROOF. Let $A^i = \{a_1^i, \cdots, a_n^i\}$ $(1 \leqq i \leqq k)$ be the k independent sets of vertices of G. If k is odd, for each j, $1 \leqq j \leqq n$, we totally color the complete subgraph B_j of G spanned by the vertices $\{a_j^1, \cdots a_j^k\}$ by k colors, with $\chi(a_j^i) = i$ for all $1 \leqq i \leqq k$. Such a coloring exists, as was shown in [1]. The graph H obtained by removing the edges thus colored satisfies $v(H) = (k-1)(n-1)$, hence by Vizing's theorem $\varepsilon(H) \leqq (k-1)(n-1) + 1$ and $\tau(G) \leqq \varepsilon(H) + k \leqq v(G) + 2$.

To prove the theorem we use induction on k. We assume that if G is a complete balanced k-partite graph then there exists a total coloration of G by $v(G) + 2$ colors, using only k colors for the vertices. Assume now that the theorem is proved for $k < k_0$. If k_0 is odd then by the preceding first part of the proof, G admits a total coloring as required. If k_0 is even, put $k_0 = 2r$. Let H_1 be the r-partite subgraph of G spanned by $\bigcup_{i=1}^r A^i$ and let H_2 be the r-partite subgraph of G spanned by $\cup_{i=r+1}^{k_0} A^i$. Obviously, H_1 and H_2 are isomorphic. By the induction hypothesis $\tau(H_i) \leqq v(H_i) + 2$ $i = 1, 2$, and in the total coloring of H_i only r colors are used to color the vertices. We may assume that $n \geqq 2$ (otherwise G is a complete graph), hence $n(r-1) + 2 \geqq 2r$ and therefore, by a permutation of the colors if necessary, we may assume that $H_1 \cup H_2$ is totally colored by $n(r-1) + 2$ colors, vertices belonging to distinct independent sets having distinct colors, and $2r$ colors being used to color the vertices. The graph obtained from G by removing the edges thus colored is a complete nr-by-nr bipartite graph. Since this graph is nr-edge-colorable we obtain

$$\tau(G) \leqq n(r-1) + 2 + nr = v(G) + 2.$$

THEOREM 3. *If G is a graph with $v(G) \leqq 3$, then $\tau(G) \leqq 5$.*

PROOF. The proof proceeds by induction on the number of vertices in G, the theorem being obvious for graphs with a small number of vertices. It is easy to see that if a graph has a 2-valent vertex and if the omission of that vertex leads to a totally-5-colorable graph, then the original graph is totally-5-colorable. Hence G may be assumed cubic, and by the above, to be a graph without multiple edges.

If g, g' are two distinct vertices in a graph G, each of valence at most two, we will denote by $G(g, g')$ the graph obtained from G by adjoining to G a new vertex h and the edges gh and $g'h$.

A *circuit in* G will be called *admissible* if it does not contain a diagonal and if the vertices forming the circuit are $\{h_1, \cdots, h_m\}$ and for each i the vertex joined by an edge to h_i in $G \backslash \{h_1, \cdots, h_m\}$ is g_i, then $g_i \neq g_j$ if $i \neq j$.

Assume first that G contains an admissible circuit and let G' be the graph obtained from G by removing the vertices $\{h_1, \cdots, h_m\}$ and the edges incident with them. By the induction hypothesis, $\tau(G') \leqq 5$. Without loss of generality, we may assume that in the given total coloring of G', g_1 and the two edges incident with it are colored as shown in Fig. 1.

Case 1. g_2 and the two edges incident with it *are not* colored in G' as indicated in Fig. 2 with $\chi(g_2) \neq 1$.

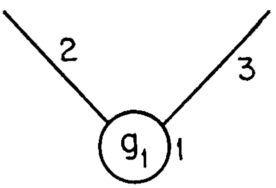

Fig. 1

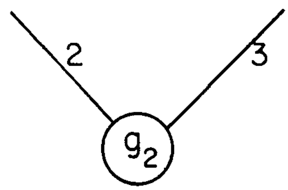

Fig. 2

Let G_2 be the graph obtained from G' by adjoining to it the configuration shown in Fig. 3.

We will show that G_2 can be totally colored by 5 colors with $\chi(h_1) = 2$, $\chi(h_1 h_2) = 1$ and $\chi(h_2) \neq 3$. Indeed the following table covers all possibilities:

$\chi(E_1)$	$\chi(E_2)$	$\chi(g_2)$	$\chi(g_2 h_2)$	$\chi(h_2)$
		3	$[\chi(E_1), \chi(E_2), 1, 3]$	$[\chi(g_2 h_2), 1, 2, 3]$
$\neq 3$	$\neq 3$	$\neq 3$	3	$[\chi(g_2), 1, 2, 3]$
3	$\neq 2$	2	2	$[\chi(g_2), 1, 2, 3]$
3	$\neq 2$	2	$[\chi(E_2), 1, 2, 3]$	$[\chi(g_2 h_2), 1, 2, 3]$
3	$\neq 2$	1(by assumption)	4	5

By $[a, b, c, d]$ we mean a color distinct from the colors $\{a, b, c, d\}$.

Let $G_i = G_{i-1}(h_{i-1}, g_i)$ $(3 \leqq i \leqq m)$ where h_i is the vertex adjoined to G_{i-1} to obtain G_i. Since a total coloring of G_i by five colors can be obtained by extending a total coloring of G_{i-1}, we have: $\tau(G_m) \leqq 5$, and the colors of the vertices and edges in Fig. 3 are as described above. We now adjoin the edge hh_m in G_m (thus obtaining G) and show that $\tau(G) \leqq 5$.

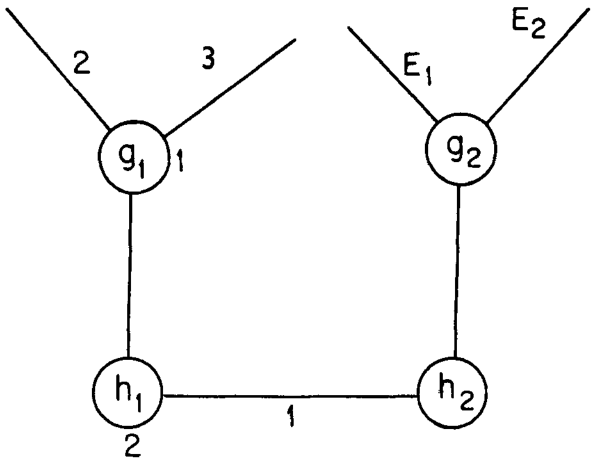

Fig. 3

Assume first that in G_m, $\chi(h_m) \neq 2$; if $\{\chi(h_m), \chi(h_m g_m), \chi(h_{m-1} h_m)\} \neq \{3,4,5\}$, we color in G $\chi(h_1 h_m) = [1,2,\chi(h_m), \chi(h_m g_m), \chi(h_{m-1} h_m)]$ and $\chi(h_1 g_1) = [1,2,3,\chi(h_1 h_m)]$. If $\{\chi(h_m), \chi(h_m g_m), \chi(h_m h_{m-1})\} = \{3,4,5\}$ and $\chi(h_m) \neq 3$ we color $\chi(h_1) = 3$, $\chi(h_1 h_m) = 2$ and $\chi(h_1 g_1) = 4$, if $\chi(h_m) = 3$ let $\chi(h_1) = [1,2,3,\chi(h_2)]$, $\chi(h_1 h_m) = 2$ and $\chi(h_1 g_1) = [1,2,3,\chi(h_1)]$. In all these cases we obtain a total coloring of G by five colors. Hence we may assume now that

$\chi(h_m) = 2$. If $\chi(h_m g_m) = 3$ or $\chi(h_m h_{m-1}) = 3$ let $\chi(h_1) = 3$, $\chi(h_m h_1)$

$$= [1,2,3,\chi(h_m g_m), \chi(h_m h_{m-1})] \text{ and } \chi(h_1 g_1) = [1,2,3,\chi(h_1 h_m)].$$

Finally, if $\chi(h_m g_m) \neq 3$ and $\chi(h_m h_{m-1}) \neq 3$, let $\chi(h_1) = [1,2,3,\chi(h_2)]$ $\chi(h_m h_1) = 3$ and $\chi(h_1 g_1) = [1,2,3,\chi(h_1)]$. This completes the proof of Case 1. We may assume now the second case, namely that no pair of adjacent vertices on the admissible circuit satisfies the coloring relations described above. This means that for every g_i, the pair of edges incident with it are colored 2 and 3 respectively, and $\chi(g_i) \neq \chi(g_{i+1})$ for $1 \leqq i \leqq m$ $(g_{m+1} = g_1)$. Without loss of generality we may assume that $\chi(g_2) = 4$. Let $\chi(h_2) = 3$, $\chi(h_1) = 4$, $\chi(h_1 h_2) = 1$ and $\chi(h_2 h_3) = 2$. The rest of the edges and vertices in G will be colored as follows:

$$\chi(h_k) = \chi(g_{k+1}),\ 2 \leqq k \leqq m-1,\ \chi(h_k h_{k+1}) = 2 \text{ or } 3$$

(alternatingly). Obviously, by our assumption this coloring is admissible thus far and furthermore, on every edge $(g_i h_i)$ $1 \leqq i \leqq m$, we have only four constraints and therefore this coloring can be extended to a total coloring of G by five colors.

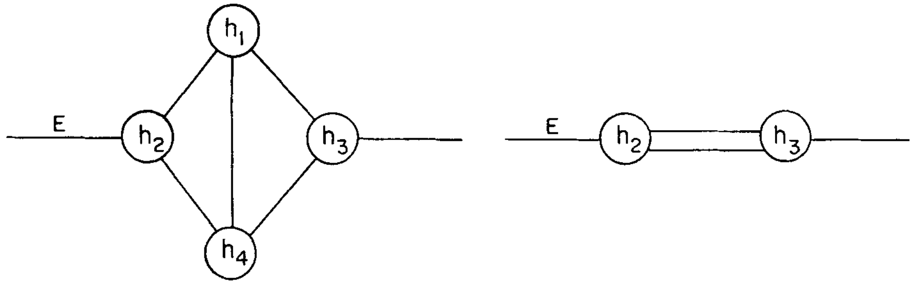

Fig. 4a

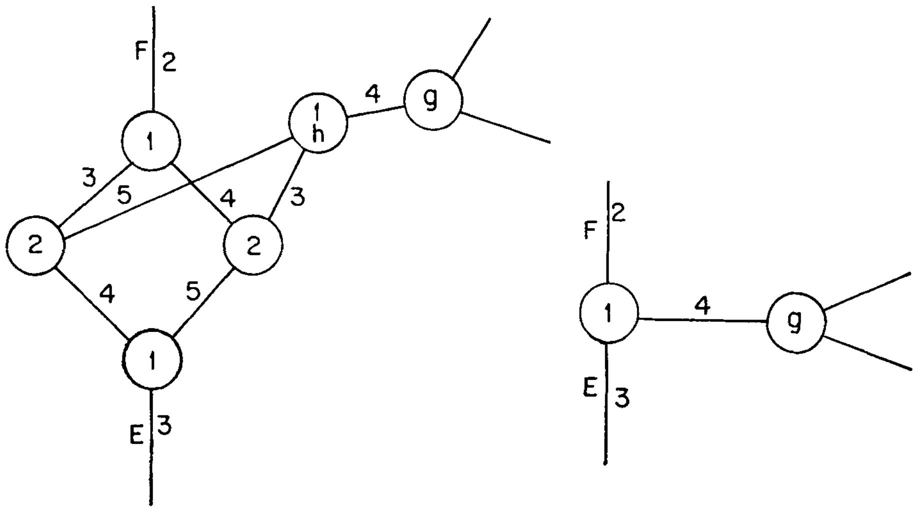

Fig. 4b

If the shortest circuit in G is of length $\geqq 5$ then G contains an admissible circuit (a shortest circuit must be admissible). Hence if G does not contain an admissible circuit it must contain a non-admissible triangle, Fig. 4a, or a non-admissible quadrangle, Fig. 4b. In Fig. 4a we identify the vertex h_1 with h_3 and h_2 with h_4, obtaining a graph G' with $v(G') = 3$ having two vertices fewer than G. Hence by the induction hypothesis $\tau(G') \leqq 5$. Without loss of generality we may assume that in a total coloring of G' by 5 colors the section shown in Fig. 4a has $\chi(h_2) = 2$, $\chi(h_3) = 1$, $\chi(h_2h_3) = \{3,4\}$ and $\chi(E) = 1$ or 5. Going back to G we color: $\chi(h_3) = 1$, $\chi(h_2) = 2$, $\chi(h_1) = 4$, $\chi(h_4) = 5$, $\chi(h_1h_3) = 3$, $\chi(h_1h_4) = 2$, $\chi(h_1h_2) = 1$ if $\chi(E) = 5$ and 5 if $\chi(E) = 1$. $\chi(h_4h_2) = 3$ and $\chi(h_4h_3) = 3$. Obviously this coloring yields a total coloration of G by 5 colors.

Finally, if G contains a nonadmissible 4-circuit, Fig. 4b, we identify the vertices of the 4-circuit and the vertex h as indicated in Fig. 4b, obtaining a graph G' with

$\nu(G') = 3$. Again by the induction hypothesis, $\nu(G') \leqq 5$. Assuming that in a given coloration of G' the colors are as indicated in Fig. 4b we color the vertices of the 4-circuit as indicated. Since this yields a total coloration of G by 5 colors the proof is complete.

References

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